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# Computing the Edge Irregularity Strength of Some Classes of Grid Graphs

Shahzad, M.[1,2], Hasni, R.*[2], Tarawneh, I.[3], and Asim, M. A.[4]

[1] *Faculty of Computing Sciences, Gulf College, Muscat, Oman*
[2] *Special Interest Group of Modeling and Data Analytics (SIGMDA), Faculty of Computer Science and Mathematics, Universiti Malaysia Terengganu, 21030 Kuala Nerus, Terengganu, Malaysia*
[3] *Khalid Ibn Al-Walid, Karak, Jordan*
[4] *Division of Computing, Analytics and Mathematics, School of Science and Engineering, University of Missouri-Kansas City, MO 64110, USA*

*E-mail: hroslan@umt.edu.my*
**Corresponding author*

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## Abstract

Let $G$ be a simple graph. A function $\phi : V(G) \to \{1, 2, \ldots, k\}$ a vertex $k$-labeling which assigns labels to the vertices of $G$. For any edge $xy$ in $G$, we define the weight of this edge as $w_\phi(xy) = \phi(x)+\phi(y)$. If all the edge weights are distinct, then $\phi$ is termed as an edge irregular $k$-labeling of $G$. The smallest possible value of $k$ for which the graph $G$ possesses an edge irregular $k$-labeling is denoted as the edge irregularity strength of $G$ and is represented as $es(G)$. In this paper, we investigate the edge irregular $k$-labeling of some classes of grid graphs, namely rhombic graph $R_n^m$, triangular graph $L_n^m$ and octagonal graph $O_n^m$. As by-product, we obtain their precise value of edge irregularity strength.

**Keywords:** rhombic grid; triangular grid; octagonal grid; edge irregular $k$-labeling; edge irregularity strength.

## 1 Introduction

Consider a simple connected graph $G = (V, E)$ with a vertex set denoted as $V(G)$ and an edge set as $E(G)$. In the realm of graph theory, graph labeling is a fundamental method used to assign positive integer labels or weights to various elements of a graph, including vertices, edges, or both. This technique holds significant importance and finds widespread practical applications, encompassing a diverse range of scenarios where it aids in modeling, analysis, and problem-solving. It proves indispensable in data analysis, streamlining data clustering, and facilitating machine learning tasks. Notably, it is employed in routing and path planning, image processing, bioinformatics, and social network analysis. Moreover, graph labeling extends its utility to fields such as chemistry, optimization problems, code generation, game theory, and semantic web annotation. Its versatility and adaptability render it a valuable instrument for modeling, analyzing, and optimizing complex systems and networks. The irregular labeling or networks are mostly used in the analysis of networks. What happens when networks are completely irregular, or what happens when they are completely regular? One can study and analyze networks in this regard. Since most real-world networks are in between, we have identified both extremes. One can use this to estimate what happens within real-world networks.

Chartrand et al. [8] introduced the concept of edge $k$-labeling, denoted as $\phi$, for a graph $G$ in a way that ensures distinct edge weights, i.e., $w_\phi(x) \neq w_\phi(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. These labelings were termed "irregular assignments". The irregularity strength, denoted as $s(G)$, of a graph $G$ is defined as the smallest value of $k$ for which $G$ can have an irregular assignment using labels up to $k$. This parameter has garnered significant attention [7, 11, 12].

The concept of an edge irregular $k$-labeling for a graph $G$ was first presented by Ahmad et al. [2]. This is a vertex labeling $\phi : V(G) \to \{1, 2, \ldots, k\}$, where the edge weights $w_\phi(vu) = \phi(v)+\phi(u)$ are unique for each edge in the graph. The *edge irregularity strength* of $G$ is the lowest value of $k$ for which such an edge irregular $k$-labeling exists; it is denoted by $es(G)$. Last couple of years studies have been carried out on $es(G)$ for different families of graphs and trees [1, 4, 15]. Sometimes, mathematical approaches are hard or impossible to provide the solution. In such cases, algorithmic approaches can also be used, and recently, a lot of work has been done using algorithmic approaches. In [3], the authors computed the edge irregularity strength of bipartite graphs and wheel related graphs. Asim et al. [5, 6] used an iterated algorithm for computing the irregularity strength of complete graphs and circulant graphs, respectively. Tarawneh et al. [14] investigated the edge irregularity strength of disjoint union of certain graph. Algorithmic approaches have been used for solving graph problems efficiently. Graph algorithms are famous and have been used in different real-time applications like path determination, network flow optimization, natural language processing and machine learning models. Algorithms were used in the field of graph labeling for the first time in 2018 by Asim et al. [5] for updating upper-bound for vertex $k$-labeling of complete graph $es(K_n)$. Ahmad and colleagues conducted a computer-based experiment, as described in their paper [1], to achieve vertex $k$-labeling for complete $m$-ary trees using algorithmic methods. Subsequently, they applied this algorithmic approach to determine vertex $k$-labeling for various graph types, including wheel-related graphs, bipartite graphs, and circulant graphs, as indicated in references [3] and [6]. These innovative solutions have broadened the horizons for computer experts, offering valuable tools and insights in the field of graph labeling, as emphasized by references [4] and [14]. Specifically, a lower bound on the vertex $k$-labeling of a graph $G$ is established by the following theorem [2].

The following theorem establishes a lower bound for the vertex $k$-labeling $es$ of any graph $G$.

**Theorem 1.1.** [2] *Let $G = (V, E)$ be a simple graph with maximum degree $\Delta = \Delta(G)$. Then,*

$$es(G) \geq \max\left\{\left\lceil \frac{|E(G)|+1}{2} \right\rceil, \Delta(G)\right\}.$$

The authors in [2] established constraints on the parameter $es(G)$ and provided specific values for vertex $k$-labeling in several graph families, such as the $n \times m$ grid graph, formed by the Cartesian product of two paths. In addition, the work conducted by Tarawneh and their research team, as referenced in their paper [16], stands out for its achievement in finding the exact vertex $k$-labeling for specific types of graphs. These include the triangular graph, the zigzag graph and the Cartesian product of three paths $P_n$, $P_m$ and $P_2$. Please refer to [9, 17] and its references for additional results. Their findings have significantly contributed to the understanding of graph labeling and have practical implications in various domains. In continuation of this research, our paper focuses on determining the precise value of vertex $k$-labeling for grid graphs with distinct geometries, including rhombic, triangular, and octagonal structures. By extending the exploration of exact vertex $k$-labeling to these specific graph types, we aim to enhance our understanding of labeling in diverse grid graph contexts and its applications in real-world problem-solving and analysis.

## 2 Rhombic Grid Graph

A rhombic grid graph with the vertex set $V(R_n^m)$ and edge set $E(R_n^m)$ is denoted by $R_n^m$. Note that $|E(R_n^m)| = 4mn$ and $|V(R_n^m)| = 2mn + m + n$, where $n, m \geq 2$. The inequality $es(G) \geq \max\left\{\left\lceil \frac{|E(G)|+1}{2} \right\rceil, \Delta(G)\right\}$ was proven. Given that $\Delta(G) = 4$, $es(G) \geq \left\lceil \frac{|E(G)|+1}{2} \right\rceil = 2mn + 1$, according to Theorem 1.1. An edge irregular $2mn + 1$-labeling for $R_n^m$ is described in order to demonstrate that $2mn + 1$ is an upper bound for the $es(R_n^m)$.

$$V(R_n^m) = \{v_p^q | 1 \leq p \leq n, 1 \leq q \leq m, \} \cup \{u_p^q | 1 \leq p \leq n+1, 1 \leq q \leq m\},$$

and

$$\begin{aligned} E(R_n^m) = &\{v_p^q u_p^q | 1 \leq p \leq n, 1 \leq q \leq m\} \cup \{v_p^q u_{p+1}^q | 1 \leq p \leq n+1, 1 \leq q \leq m\} \\ &\cup \{u_p^q v_p^{q+1} | 1 \leq p \leq n, 1 \leq q \leq m\} \cup \{u_p^q v_{p-1}^{q+1} | 1 \leq p \leq n, 1 \leq q \leq m\}, \end{aligned}$$

with $|V(R_n^m)| = 2mn + m + n$ and $|E(R_n^m)| = 4mn$. Figure 1 shows the rhombic grid graph $R_n^m$ where $m = 3$ and $n = 4$.

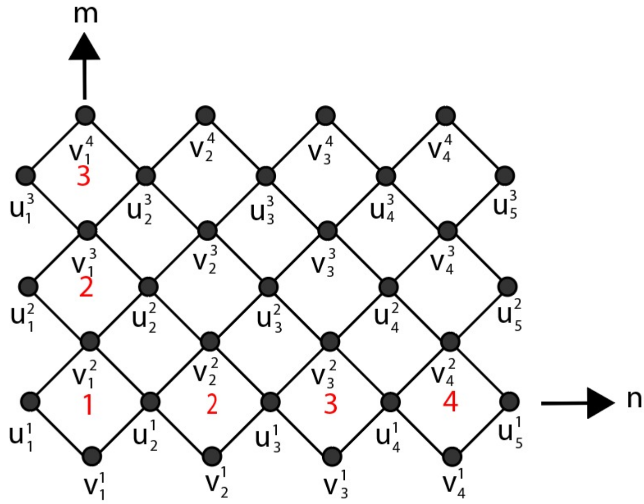


Figure 1: Rhombic grid graph $R_n^m = R_4^3$.

Now, we present our main theorem.

**Theorem 2.1.** *For* $m, n \geq 2$, $es(R_n^m) = 2mn + 1$.

*Proof.* Consider the rhombic grid graph denoted as $R_n^m$ with $V(R_n^m)$ as vertex set and $E(R_n^m)$ as edge set. Notably, $|V(R_n^m)| = 2mn+n+m$, and $|E(R_n^m)| = 4nm$. In the provided analysis, we direct our attention to rhombic grid graphs, specifically denoted as $R_n^m$ with $n, m \geq 2$. These graphs represent an array of rhombuses, with $n$ representing the number of rhombuses in a row and $m$ indicating the number of rhombuses in a column. To understand the lower bound for the vertex $k$-labeling $(es(G))$ of these graphs, we refer to a previously established result in graph theory. The vertex $k$-labeling of any graph, $es(G)$, is thus required to follow a lower bound determined by two factors: $\left\lceil \frac{|E(G)|+1}{2} \right\rceil$, and the maximum degree of a vertex in the graph, $\Delta(G)$. Since $\Delta(G) = 4$, $es(G) \geq \left\lceil \frac{|E(G)|+1}{2} \right\rceil = 2mn + 1$, according to Theorem 1.1.

This calculation results in a lower bound of $2mn + 1$ for the vertex $k$-labeling. Now, to demonstrate that $2mn + 1$ indeed serves as an upper bound for $es(R_n^m)$, we proceed to construct an edge irregular $2mn + 1$-labeling for the specific rhombic grid graph $R_n^m$. Such a labeling ensures that the weights assigned to the edges within this graph are distinct, thereby confirming that the edge irregularity strength of $R_n^m$ does not exceed $2mn + 1$. Let the vertex labeling $\phi_1 : V(R_n^m) \to \{1, 2, \ldots, 2m + 1\}$ defined as follows:

$$\phi_1(v_p^q) = \begin{cases} 1, & \text{if} \quad p = 1,\ q = 1, \\ 2p, & \text{if} \quad q = 1,\ 2 \leq p \leq n, \\ 2n(q-2) + 2p + 1, & \text{if} \quad 2 \leq q \leq m+1,\ 1 \leq p \leq n, \end{cases}$$

$$\phi_1(u_p^q) = \begin{cases} 1, & \text{if} \quad q = 1,\ p = 1, \\ 2(p-1), & \text{if} \quad q = 1,\ 2 \leq p \leq n+1, \\ 2nq - \frac{(1-(-1)^p)}{2}, & \text{if} \quad 2 \leq q \leq m,\ 1 \leq p \leq n+1. \end{cases}$$

The weight of the edges are as follows:

$$w_{\phi_1}(v_p^q u_p^q) = \begin{cases} 2, & \text{if} \quad q=1,\ p=1, \\ 4p-2, & \text{if} \quad q=1,\ 2 \le p \le n, \\ 4nq-4n+2p-\frac{(1-(-1)^p)}{2}+1, & \text{if} \quad 1 \le p \le n \ \text{ and } \ 2 \le q \le m, \end{cases}$$

$$w_{\phi_1}(v_p^q u_{p+1}^q) = \begin{cases} 2p+1, & \text{if} \quad p=1,\ q=1, \\ 4p, & \text{if} \quad 2 \le p \le n,\ q=1, \\ 4nq-4n+2i-\frac{(1-(-1)^p)}{2}+1, & \text{if} \quad 1 \le i \le n \ \text{ and } \ 2 \le q \le m, \end{cases}$$

$$w_{\phi_1}(u_p^q v_p^{q+1}) = \begin{cases} 2nq-2n+4, & \text{if} \quad p=1,\ q=1, \\ 2nq-2n+4p-1, & \text{if} \quad 2 \le p \le n,\ q=1, \\ 4nq-2n+2p-\frac{(1-(-1)^p)}{2}+1, & \text{if} \quad 1 \le p \le n \ \text{ and } \ 2 \le q \le m, \end{cases}$$

and

$$w_{\phi_1}(u_p^q v_{p-1}^{q+1}) = \begin{cases} 2nq-2n+4p-3, & \text{if} \quad 2 \le p \le n+1,\ q=1, \\ 4nq-2n+2p-\frac{(1-(-1)^p)}{2}-1, & \text{if} \quad 2 \le p \le n+1 \ \text{ and } \ 2 \le q \le m. \end{cases}$$

The uniqueness of all edge weights in the context of the vertex labeling $\phi_1$ signifies a significant result. It indicates that this particular vertex labeling, denoted as $\phi_1$, stands as an optimal choice for achieving edge irregularity. The fact that all edge weights are distinct strengthens its optimality. In essence, $\phi_1$ offers a highly efficient labeling scheme that allows for precise differentiation of edges within the graph. In this case, the labeling represents an optimal edge irregular $2mn+1$-labeling. This conclusion provides a strong basis for the proof's completion, demonstrating the effectiveness and optimality of $\phi_1$ in achieving edge irregularity within the graph. □

Figure 2 shows the graph $R_4^3$ which admits the 25-edge irregular labeling.

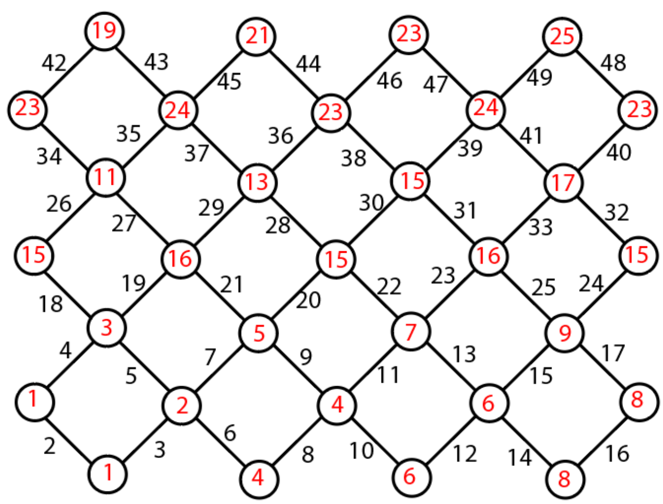

Figure 2: The edge irregularity strength of $R_4^3$ is 25.

## 3 Triangular Grid Graph

The triangular grid graph $L_n^m$ is a graph structure represented by two parameters, $n$ and $m$, where $n$ is the number of vertices in a row, and $m$ is the number of squares in a column, see Figure 3. It can be visualized as a grid composed of triangles, where each vertex represents an intersection point, and the edges correspond to the sides of these triangles. The conditions $n \geq 2$ and $m \geq 1$ indicate that the configuration is applicable when there are at least two vertices in a row and at least one square in a column.

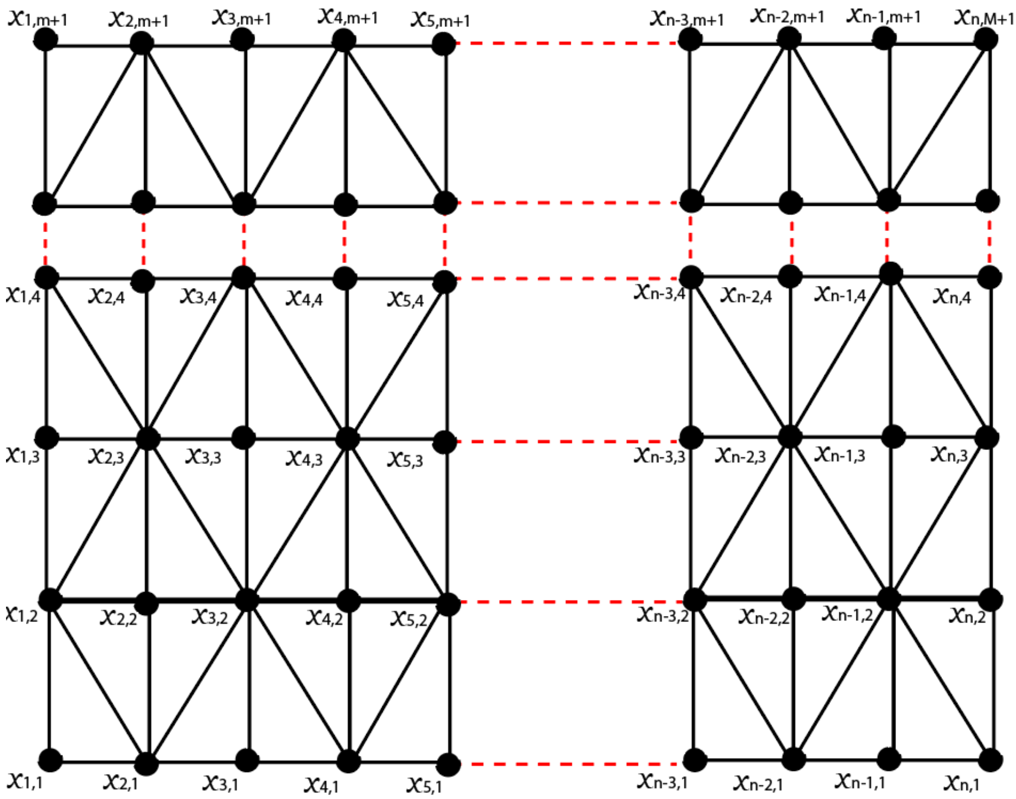

Figure 3: Triangular grid graph $L_n^m$.

The graph's vertex set, denoted as $V(L_n^m)$, represents all intersection points, while the edge set, denoted as $E(L_n^m)$, represents the connections between these points. Formally, $V(L_n^m)$ and $E(L_n^m)$ are defined as follows:

$$V(L_n^m) = \{x_{p,q} | \, 1 \leq p \leq n, 1 \leq q \leq m+1\},$$

and

$$\begin{aligned} E(L_n^m) = \{ & x_{p,q}x_{p+1,q} | \, 1 \leq p \leq n-1, 1 \leq q \leq m+1\} \\ & \cup \{x_{p,q}x_{p,q+1} | \, 1 \leq p \leq n, 1 \leq q \leq m\} \\ & \cup \{x_{p+1,q}x_{p,q+1} | \, 1 \leq p \leq n-1, 1 \leq q \leq m \text{ and } q \text{ is odd}\} \\ & \cup \{x_{p,q}x_{p+1,q+1} | \, 1 \leq p \leq n-1, 1 \leq q \leq m \text{ and } q \text{ is even}\}. \end{aligned}$$

Tarawneh et al. [16] determined the exact value of the vertex $k$-labeling for a specific type of grid graph known as the triangular grid graph, denoted as $L_n = L_n^1$. This outcome signifies his accomplishment in solving this particular graph's edge irregularity strength problem.

**Theorem 3.1.** [16] *For $n \geq 2$, $es(L_n) = 2n$.*

In this paper, we continue to investigate the exact value of the vertex $k$-labeling of $L_n^m$ where $m = 2, 3$.

**Theorem 3.2.** *For any integer $n \geq 2$, then*

$$es(L_n^2) = \begin{cases} 6, & n = 2, \\ 4n-1, & n \geq 3, \end{cases}$$

*Proof.* Consider the graph $L_n^2$ with the vertex set denoted as $V(L_n^2)$ and the edge set as $E(L_n^2)$. Notably, $|V(L_n^2)| = 3n$ and $|E(L_n^2)| = 7n - 5$. It is worth mentioning that the maximum degree of $L_n^2$, represented as $\Delta(L_n^2)$, is 6. For the special case when $n = 2$, as shown in Figure 4(a), we find that $es(L_2^2) = 6$. However, when $n \geq 3$, by invoking Theorem 1.1, we establish that $es(L_n^2) \geq \max\{\lceil \frac{7n-4}{2} \rceil, 6\} = \lceil \frac{7n-4}{2} \rceil$. Additionally, taking into account the edges $x_{p,q}$, $x_{p+1,q}$, and $x_{p,q+1}$ as parts of the entire graph $K_3$, it is clear that the minimum edge weight needs to be 3. Thus, the edge weights successfully span values in the set $\{3, 4, 5, \ldots, 4n-1\}$ under the labeling $\phi_2$. It follows from this fact that $es(L_n^2) \geq 4n - 1$.

To establish the inequality $es(L_n^2) \leq 4n - 1$, we introduce a vertex labeling $\phi_2 : V(L_n^2) \to \{1, 2, \ldots, 4n-1\}$ in the following manner:

$$\phi_2(x_{p,q}) = 4(p-1) + q, \text{ if } 1 \leq p \leq n, \ 1 \leq q \leq 3.$$

The edge weights of all edges are given as follows.

$$w_{\phi_2}(x_{p,q}x_{p+1,q}) = 8p - 4 + 2q, \text{ if } 1 \leq p \leq n, \ 1 \leq q \leq 3,$$

$$w_{\phi_2}(x_{p,q}x_{p,q+1}) = 8p - 7 + 2q, \text{ if } 1 \leq p \leq n, \ 1 \leq q \leq 2,$$

$$w_{\phi_2}(x_{p+1,q}x_{p,q+1}) = 8p - 1, \text{ if } 1 \leq p \leq n-1, \ q = 1,$$

and

$$w_{\phi_2}(x_{p,q}x_{p+1,q+1}) = 8p + 1, \text{ if } 1 \leq p \leq n-1, \ q = 2.$$

The vertex labeling $\phi_2$ is determined to be the optimal edge irregular labeling with $4n-1$ labels since every edge weight shows unique values. This indicates that the proof has ended here. □

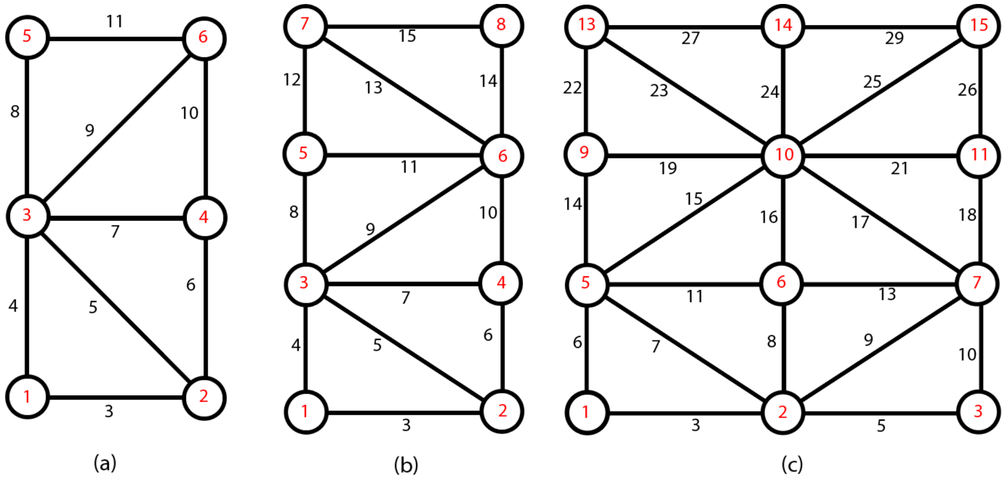


Figure 4: Triangular grid graphs of $L_2^2$,$L_2^3$, and $L_3^3$.

Figure 5 shows the graph $L_5^2$ which admits the 19-edge irregular labeling.

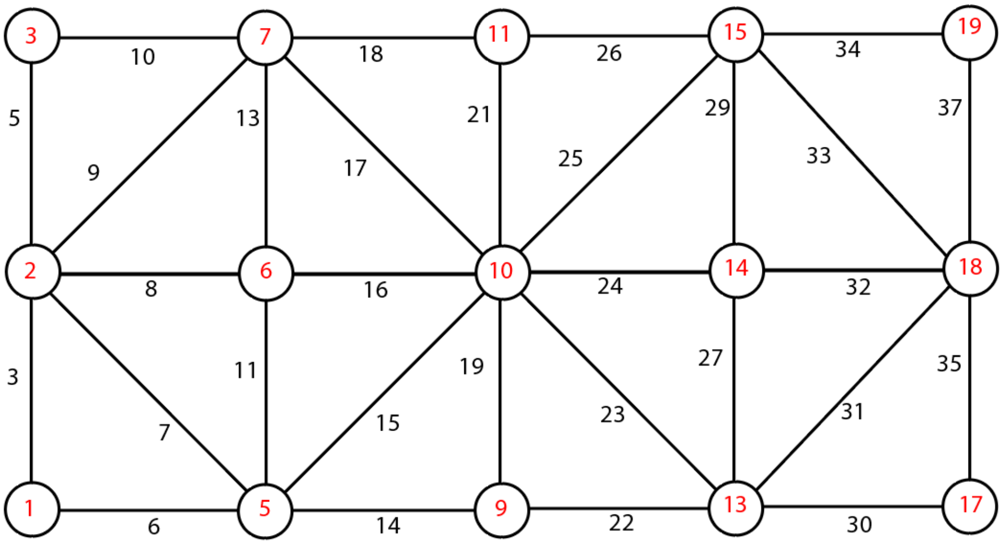

Figure 5: An edge irregular 19-labeling of octagonal grid graph $L_5^2$.

**Theorem 3.3.** *For any integer $n \geq 2$, then*

$$es(L_n^3) = \begin{cases} 7n-6, & n \in \{2,3\}, \\ 6n-2, & n \geq 4. \end{cases}$$

*Proof.* Let $L_n^3$ represents a graph with vertex set $V(L_n^3)$ and edge set $E(L_n^3)$. Note that $|V(L_n^3)| = 4n$ and $|E(L_n^3)| = 10n - 7$. The maximum degree of $L_n^3$ is 6 as depicted in Figure 3. Firstly, graph $L_n^3$ with $n = 2, 3$ is shown in Figure 4(b) and 4(c), respectively. If $n \geq 4$, according to Theorem 1.1, we establish that $es(L_n^3) \geq \max\{\lceil \frac{10n-6}{2} \rceil, 6\} = 10n - 6$. The minimum edge weight of $K_3$ must be at least 3, as the edges $x_{p,q}$, $x_{p+1,q}$, and $x_{p,q+1}$ are components of the entire graph. As thus, the labeling $\phi_3$'s edge weights take values from the set $\{3, 4, 5, \ldots, 6n-2\}$. We build a suitable vertex labeling $\phi_3 : V(L_n^3) \to \{1, 2, \ldots, 6n-2\}$ so as to illustrate the inequality $es(L_n^3) \leq 6n - 2$.

$$\phi_3(x_{p,q}) = 6(p-1) + q, \text{ if } 1 \leq p \leq n, \ 1 \leq q \leq 4.$$

The edge weights of all edges are given as follows:

$$w_{\phi_3}(x_{p,q}x_{p+1,q}) = 12p - 6 + 2q, \text{ if } 1 \leq p \leq n-1, \ 1 \leq q \leq 4,$$

$$w_{\phi_3}(x_{p,q}x_{p,q+1}) = 12p - 11 + 2q, \text{ if } 1 \leq p \leq n, \ 1 \leq q \leq 3,$$

$$w_{\phi_3}(x_{p+1,q}x_{p,q+1}) = 12p + 2q - 5, \text{ if } 1 \leq p \leq n-1, \ 1 \leq q \leq 4, \ q \text{ is odd},$$

and

$$w_{\phi_3}(x_{p,q}x_{p+1,q+1}) = 12p + 2q - 5, \text{ if } 1 \leq p \leq n-1, \ 2 \leq q \leq 4, \ q \text{ is even}.$$

As all edge weights are distinct, the vertex labeling $\phi_3$ is an optimal edge irregular labeling with $(6n-2)$ labels. This concludes the proof. □

Figure 6 shows the graph $L_6^3$ which admits the 34-edge irregular labeling.

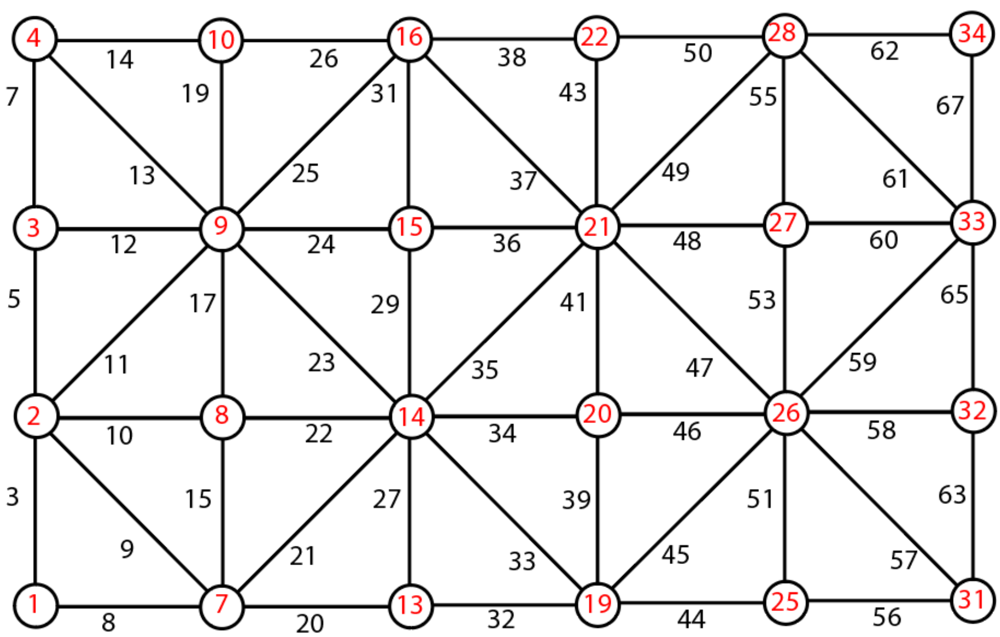


Figure 6: An edge irregular 34-labeling of $L_6^3$.

We are not able find the exact value of edge irregular labeling for $L_m^n$ for general $n \geq 2$, $m \geq 1$. Therefore, we re-state the following open problem (see also [16]).

**Problem 1.** Determine the exact value of $es(L_m^n)$ for $n \geq 1$ and $m \geq 4$.

## 4 Octagonal Grid Graph

In the research conducted by Siddiqui and their team, as referenced in their paper [13], they determined the exact value of what is known as the total edge irregularity strength for the octagonal grid graph. This value quantifies how irregular the edges are labeled in this specific type of graph.

Additionally, in another study mentioned in a separate paper [10], different authors calculated the exact value of what's referred to as the edge $H$-irregularity strength for both hexagonal and octagonal grid graphs. This measure helps us understand the irregularity of edges in a different context, where "$H$" presumably signifies a specific type of irregularity.

This section of the current work is dedicated to further investigating and determining the exact value of the vertex $k$-labeling for the octagonal grid graph. This research extends the understanding of how edges are labeled irregularly in the context of this particular type of graph, contributing to the broader field of graph theory.

We work with finite graphs. For $m$ and $n$, both greater than or equal to 1 (i.e., $m, n \geq 1$), we represent the octagonal grid graph as $O_n^m$. This graph is illustrated in Figure 7, creating a planar map consisting of $m$ rows and $n$ columns of octagons.

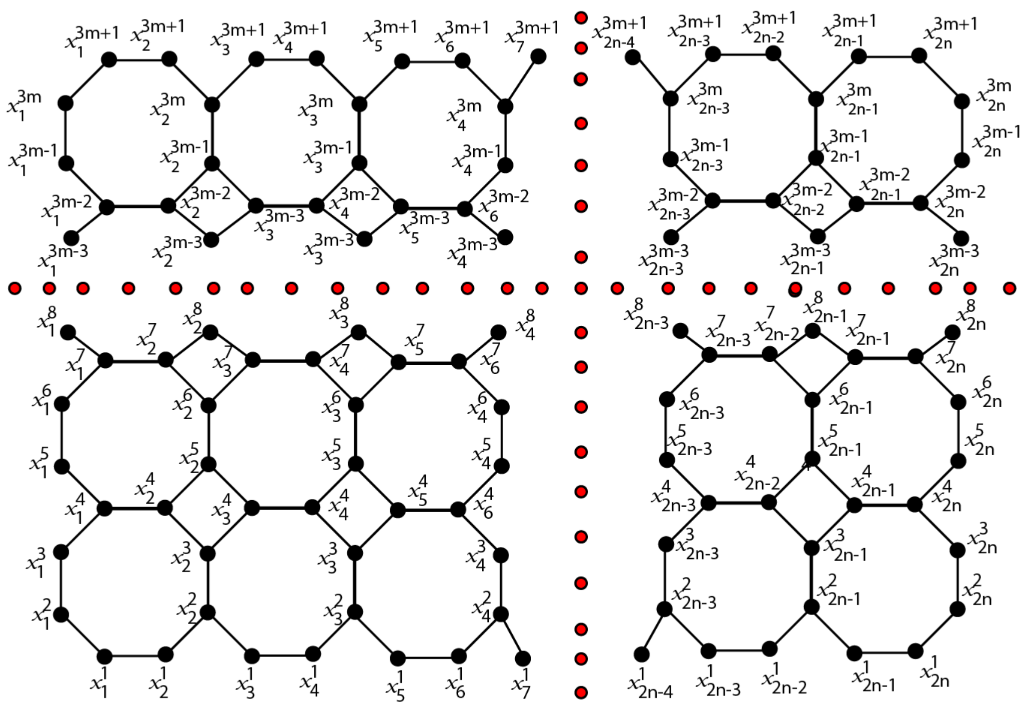


Figure 7: The octagonal grid $O_n^m$.

To reference its components, we use the notations $V(O_n^m)$ for the vertex set and $E(O_n^m)$ for the edge set. This simplifies the definition of the graph's structure and properties.

$$V(O_n^1) = \{x_p^1 | \, 1 \leq p \leq 2n-1, p \text{ odd } \} \cup \{x_p^1 | \, 1 \leq p \leq 2n, p \text{ even } \} \cup \{x_{2n}^2, x_{2n}^3 \},$$

$$E(O_n^1) = \{x_p^1x_{p+1}^1|\, 1 \le p \le 2n-1, p\text{ odd}\} \cup \{x_p^2x_{2p-2}^1|\, 2 \le p \le n\} \cup \{x_p^2x_{2p-2}^1|\, 2 \le p \le n\}$$
$$\cup \{x_p^2x_p^3;\, 1 \le p \le n+1,\} \cup \{x_p^4x_{p+1}^4|\, 1 \le p \le 2n-1, p\text{ is odd}\} \cup \{x_p^3x_{2p-1}^4|\, 1 \le p \le n\}$$
$$\cup \{x_{p+1}^3x_{2p-2}^4|\, 1 \le p \le n+1\},$$

with $|V(O_n^m)|$ =$(4m+2)n+2m$ and $|E(O_n^m)|$= $(6m+1)n+m$.

We determine the precise value of the vertex $k$-labeling $(es)$ for the octagonal grid graph $O_n^1$, where $n \ge 1$, in the following theorem. In particular, we find $es(O_n^1)$, which represents the vertex $k$-labeling of the octagonal grid graph with order $n \ge 1$.

**Theorem 4.1.** *For any integer $n \ge 2$, then $es(O_n^1) = \lceil \frac{7n+2}{2} \rceil$.*

*Proof.* Let $O_n^1$ represents a graph with vertex set $V(O_n^1)$ and edge set $E(O_n^1)$. Note that $|V(O_n^1)| = 6n+2$ and $|E(O_n^1)| = 7n+1$. We establish that $es(O_n^1) \ge \max\{\lceil \frac{7n+1+1}{2} \rceil\} = \lceil \frac{7n+2}{2} \rceil$. Thus, the edge weight under the labeling $\phi_4$ attain values $\{2,3,4,\dots,7n+2\}$. To prove the inequality $es(O_n^1) \le \lceil \frac{7n+2}{2} \rceil$, we establish appropriate vertex labeling $\phi_4$ :$V(O_n^1) \to \{1,2,\dots,\lceil \frac{7n+2}{2} \rceil\}$ such that

$$\phi_4(x_p^1) = \begin{cases} 7\left(\frac{p-4}{4}\right)+7, & \text{if} \quad p \equiv 0(mod\,4), \\ 7\left(\frac{p-1}{4}\right)+3, & \text{if} \quad p \equiv 1(mod\,4), \\ 7\left(\frac{p-2}{4}\right)+3, & \text{if} \quad p \equiv 2(mod\,4), \\ 7\left(\frac{p-3}{4}\right)+6, & \text{if} \quad p \equiv 3(mod\,4), \end{cases}$$

$$\phi_4(x_p^2) = \begin{cases} 7\left(\frac{p-1}{2}\right)+1, & \text{if} \quad p \text{ is odd}, \\ 7\left(\frac{p-2}{2}\right)+5, & \text{if} \quad p \text{ is even}, \end{cases}$$

$$\phi_4(x_p^3) = \begin{cases} 7\left(\frac{p-1}{2}\right)+1, & \text{if} \quad p \text{ is odd}, \\ 7\left(\frac{p-2}{2}\right)+4, & \text{if} \quad p \text{ is even}, \end{cases}$$

$$\phi_4(x_p^4) = \begin{cases} 7\left(\frac{p-1}{4}\right)+2, & \text{if} \quad p \equiv 1(mod\,4), \\ 7\left(\frac{p-2}{4}\right)+3, & \text{if} \quad p \equiv 2(mod\,4), \\ 7\left(\frac{p-3}{4}\right)+6, & \text{if} \quad p \equiv 3(mod\,4), \\ 7\left(\frac{p-4}{4}\right)+6, & \text{if} \quad p \equiv 0(mod\,4). \end{cases}$$

The weight of all the edges are as follows:

$$\begin{aligned} \phi_4(x_p^1x_{p+1}^1) &= 7(\frac{p-1}{2})+6, \quad \text{if} \quad p \text{ is odd, } 1 \le i \le 2n-1, \\ \phi_4(x_p^2x_{2p-1}^1) &= 7p-3, \quad \text{if} \quad 1 \le p \le n, \\ \phi_4(x_p^2x_{2p-2}^1) &= 7p-6, \quad \text{if} \quad 2 \le p \le n, \\ \phi_4(x_p^2x_p^3) &= 7p-5, \quad \text{if} \quad 1 \le p \le n+1, \\ \phi_4(x_p^4x_{p+1}^4) &= 7p-2, \quad \text{if} \quad p \text{ is odd, } 1 \le p \le 2n-1, \\ \phi_4(x_p^3x_{2p-1}^4) &= 7p-4, \quad \text{if} \quad 1 \le p \le n, \\ \phi_4(x_{p+1}^3x_{2p-2}^4) &= 7p, \quad \text{if} \quad 1 \le p \le n+1. \end{aligned}$$

The vertex labeling $\phi_4$ is an optimal edge irregular $\lceil \frac{7n+2}{2} \rceil$-labeling because all edge weights are different. The proof is now complete. □

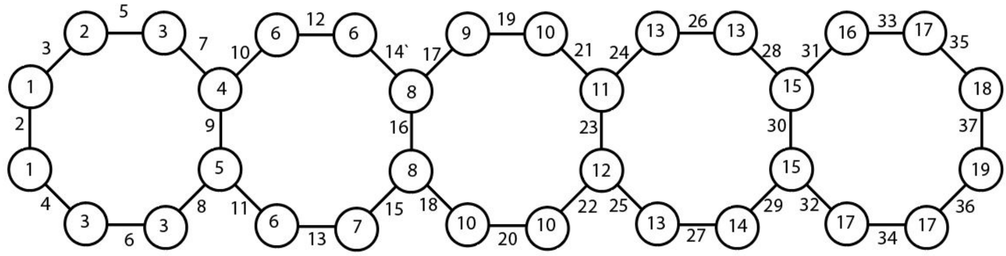

Figure 8: An edge irregular 19-labeling of $O_5^1$.

Figure 8 shows the graph $O_5^1$ which admits the 19-edge irregular labeling.

Our attempts to ascertain the exact value of edge irregular labeling for $O_m^n$ in the context of all generic $n \geq 2$ and $m \geq 2$ have not yielded success. This remains an unsolved challenge, presenting an open problem for researchers. The exploration of edge irregular labeling in the specific mathematical structure $O_m^n$ proves elusive, underscoring the complexity of the problem. This enigma beckons to those in the mathematical community to unravel its intricacies and contribute to the advancement of knowledge in this domain. The quest for understanding the nature of edge irregular labeling in $O_m^n$ stands as an intriguing mathematical puzzle, awaiting fresh perspectives and innovative solutions from aspiring researchers.

To conclude this section, we suggest the problem as given below.

**Problem 2.** Investigate the precise value of $es(O_m^n)$ for all $n \geq 2$ and $m \geq 2$.

## 5 Conclusion

The paper discusses the concept of edge $k$-labeling in graph theory, focusing on irregular assignments and the edge irregularity strength of graphs. It references several studies that explore different families of graphs and trees, as well as the application of algorithmic approaches in determining edge irregularity strength. Theorems are presented to establish lower bounds for the edge irregularity strength of certain graph families, including rhombic, triangular, and octagonal grid graphs.

For rhombic grid graphs $R_n^m$, the paper proves that $es(R_n^m) = 2mn+1$ for $m, n \geq 2$. It provides a detailed proof and construction of an optimal edge irregular labeling for these graphs. Next, for triangular grid graphs $L_n^m$, where $m = 2, 3$ and $n \geq 2$, the paper determines the exact edge irregularity strength. For $L_n^2$, the edge irregularity strength is shown to be $6$ for $n = 2$ and $4n - 1$ for $n \geq 3$. Similarly, for $L_n^3$, the edge irregularity strength is $7n-6$ for $n = 2, 3$ and $6n-2$ for $n \geq 4$. The proofs involve establishing lower bounds using existing theorems and constructing optimal edge irregular labeling.

The paper contributes to the understanding of edge irregularity strength in various graph families and provides precise values for specific cases, enhancing the theoretical foundation of graph labeling and its practical applications.

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**Conflicts of Interest** The authors declare no conflict of interest.

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